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Mixing Angle and Glashow Algebra

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Considering transformations in the basis of fundamental fields on a principal fiber bundle, without modification in the space-time sector, we construct an algebra *GA*, which we call Glashow algebra. The structure constants of this algebra depend on a mixing angle. The Lagrangian of the gauge theory of electroweak interactions without masses is obtained using a representation of *GA* which is the transformed of the adjoint representation of $SU(2) \otimes U(1)$, and does not coincide with the adjoint representation of *GA*. The mixing angle is automatically present in the theory if *GA* is used.

1. INTRODUCTION

All the fundamental interactions, with the exception of gravitation, are described by gauge theories. Such is the case for the well-established QED, for the Standard Model, and for QCD. From the geometrical point of view such theories correspond, in general terms, to the construction of a principal fiber bundle with the corresponding gauge group as the structure group and space-time as the base manifold (Aldrovandi and Pereira, 1995). The choice of a connection, which establishes the unicity of decomposition of any vector field on the space tangent to the bundle into a vertical and a horizontal component, corresponds to the introduction of a gauge potential. The well-known transformation of connections by the action of the group (adjoint representation), when pulled back to space-time, gives rise to the usual transformation of gauge fields. In each associated fiber bundle the covariant derivative defined in the principal fiber bundle will acquire a form corresponding to the representation of the fields to which it is being applied. That form is exactly what comes up when the interactions are introduced through the minimal coupling prescription.

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In consequence, a gauge theory can be locally described on the bundle by the following commutation relations (Cho, 1975)

$$\begin{aligned} [D_\mu, D_\nu] &= -F^a{}_{\mu\nu} X_a \\ [D_\mu, X_a] &= 0 \\ [X_a, X_b] &= f^c{}_{ab} X_c \end{aligned} \tag{1}$$

where the X_a are the fundamental vector fields on bundle, which represent the generators of the algebra of the structure group, and D_μ is the covariant derivative given by

$$D_\mu = \partial_\mu - gA^a{}_\mu X_a \tag{2}$$

with $A^a{}_\mu$ being the gauge field, which is a connection, and g is a coupling constant. $F^a{}_{\mu\nu}$ is the field strength of the gauge field,

$$F^a{}_{\mu\nu} = g[\partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + gf^a{}_{bc} A^b{}_\mu A^c{}_\nu] \tag{3}$$

Three of the four interactions of nature are described by gauge theories. For electromagnetism, the structure group is $U(1)$, an Abelian group, and the commutation relations and field strength are, respectively,

$$\begin{aligned} [D_\mu, D_\nu] &= -F_{\mu\nu} \\ [D_\mu, X_0] &= 0 \\ [X_0, X_0] &= 0 \end{aligned} \tag{4}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5}$$

where X_0 is the generator of $U(1)$.

For QCD (Cheng and Li, 1984) the structure group is $SU(3)$, which has eight generators and, in the Gell-Mann basis, the following structure constants:

$$\begin{aligned} &f^3{}_{12} = 1, \quad f^7{}_{41} = \frac{1}{2}, \quad f^6{}_{15} = -\frac{1}{2}, \quad f^6{}_{24} = \frac{1}{2}, \quad f^7{}_{25} = \tfrac{1}{2} \\ &f^5{}_{34} = \frac{1}{2}, \quad f^7{}_{36} = -\frac{1}{2}, \quad f^8{}_{45} = \frac{\sqrt{3}}{2}, \quad f^8{}_{67} = \frac{\sqrt{3}}{2} \end{aligned} \tag{6}$$

The commutation relations and field strength are directly obtained using (6) in (1) and (3).

In the case of electroweak interactions, the gauge group is not $SU(2) \otimes U(1)$ at all, because to obtain the couplings a mixing angle must be introduced. We cannot write for this theory the structure constants as we wrote above for QCD and QED.

In the present paper we examine the introduction of a mixing angle in a non-Abelian gauge theory through a modification of the algebra which makes it possible to write the structure constants in a way analogous to that of QCD and QED. This leads to a new algebra which we call *Glashow algebra* (*GA*), giving a geometrical interpretation for the introduction of the mixing angle in electroweak theory. This means that we obtain the Lie algebra corresponding to $SU(2) \otimes U(1)\big|_{mixed}$. The usual Lagrangian of gauge theories is obtained by taking the trace. We do obtain the Lagrangian of the Glashow model (electroweak interactions with no massive bosons) (Mandl and Shaw, 1984) in that way. Notice that it is not at all evident that this can be done, as the algebra *GA* is nonsemisimple.

In Section 2 we present the construction of the representations of the direct product $SU(2) \otimes U(1)$ which is needed to obtain some representations of the Glashow algebra. In Section 3, we construct the Glashow algebra and in Section 4 we obtain three representations for it. Section 5 contains the calculus of traces and the construction of the Lagrangian for the Glashow model. Section 6 is reserved for our conclusion and final remarks.

2. CONSTRUCTION OF THE REPRESENTATIONS OF THE DIRECT PRODUCT $SU(2) \otimes U(1)$

Let $\{X_a\}$ be the three generators of $SU(2)$ and X_0 that of $U(1)$. We now proceed to construct the representations associated to the direct product of $SU(2)$ and $U(1)$.

Consider the commutation relations of both algebras separately:

$$[X_a, X_b] = \epsilon^c{}_{ab} X_c \tag{7}$$

$$[X_a, X_b] = 0 \qquad \text{for} \quad a \text{ or } b = 0 \tag{8}$$

For the fundamental representation we use the condition

$$tr[(T_a)^2] = -1/2 \tag{9}$$

It follows that the fundamental representation is composed of 3×3 matrices whose squares have trace equal to $-1/2$. Condition (9) characterizes the fundamental representation of $SU(2)$ alone. We shall keep the same condition to get a representation which extends that representation. The generators are

$$T_1 = \begin{bmatrix} 0 & -i/2 & 0 \\ -i/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$T_2 = \begin{bmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

$$T_3 = \begin{bmatrix} -i/2 & 0 & 0 \\ 0 & i/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (12)$$

$$T_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i/\sqrt{2} \end{bmatrix} \quad (13)$$

All these matrices satisfy (7)–(9) and can be taken to generate the fundamental representation of the direct product $SU(2) \otimes U(1)$.

For the adjoint representation we have 4×4 matrices, whose squares have traces

$$tr[(X_a)^2] = -2 \quad (14)$$

a condition which is also valid for the adjoint representation of $SU(2)$.

We obtain the matrices

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (15)$$

$$X_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (16)$$

$$X_3 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (17)$$

$$X_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{2} \end{bmatrix} \quad (18)$$

satisfying (7), (8), and (14). They generate the adjoint representation of the direct product.

3. THE GLASHOW ALGEBRA

Suppose we have a gauge theory whose commutation relations are given by (1) and whose gauge potentials (or connections) and field strength are, respectively,

$$A_\mu = A^a{}_\mu X_a \tag{19}$$

and (3). On the fiber bundle, they transform as

$$X_a(A^b{}_\mu) = f^b{}_{ca} A^c{}_\mu \tag{20}$$

$$X_a(F^b{}_{\mu\nu}) = f^b{}_{ca} F^c{}_{\mu\nu} \tag{21}$$

For the direct product $SU(2) \otimes U(1)$, we observe that there are no charged fields in the theory. Interaction terms involve $f^b{}_{ac}$. In consequence, we see from (8) that there exist no interaction terms between the Abelian and non-Abelian sector. From the experimental data on electroweak interactions we know that there are two charged bosons W^+ and W^- and that there is a mixture between the Abelian and non-Abelian sectors, which gives an essential contribution to the cross section of the scattering (Mandl and Shaw, 1984; Greiner and Müller, 1996)

$$e^+ + e^- \to W^+ + W^- \tag{22}$$

In the usual gauge theory for the electroweak interaction this problem is solved by introducing a mixing angle directly in the Lagrangian. The physical fields appear as mixtures of the original gauge potentials.

Our aim is to interpret the introduction of the mixing angle from the algebraic point of view. We shall obtain a usual gauge theory considering not the direct product algebra, but another algebra GA which we will construct. In order to achieve this aim we should answer the following question:

What is the set of commutation relations which corresponds to the $SU(2) \otimes U(1)|_{mixed}$?

We call $\{X_a\}$ the basis of fields corresponding to the gauge fields $A_\mu = A^a{}_\mu X_a$, which satisfy the commutation relations

$$[X_a, X_b] = f^c{}_{ab} X_c \tag{23}$$

We associate a new basis of fields $\{X'_a\}$ to the physical potentials $A'^a{}_\mu$ which include two charged and two neutral fields:

$$A'_\mu = A'^a{}_\mu X'_a \tag{24}$$

The modification has been made only in the algebraic sector. Consequently, we expect to have no change in space-time:

$$A'_\mu = A_\mu \tag{25}$$

The neutral fields are $A'^0{}_\mu$ and $A'^3{}_\mu$, while the charged fields are given by

$$A'^1{}_\mu = \frac{1}{\sqrt{2}}(A^1{}_\mu - iA^2{}_\mu) \qquad \text{or} \qquad A^1{}_\mu = \frac{1}{\sqrt{2}}(A'^1{}_\mu + A'^2{}_\mu) \tag{26}$$

$$A'^2{}_\mu = \frac{1}{\sqrt{2}}(A^1{}_\mu + iA^2{}_\mu) \qquad \text{or} \qquad A^2{}_\mu = \frac{i}{\sqrt{2}}(A'^1{}_\mu - A'^2{}_\mu) \tag{27}$$

To consider the charged fields in the direct product $SU(2) \otimes U(1)$ is not enough to produce the correct couplings between the fields. We observe in the Lagrangian or in the equation of motion that the absence of interaction between $A'^0{}_\mu$ and the other components is due to the values of the structure constants. We must therefore modify the algebra to obtain the couplings. For that we begin by making the following transformation:

$$\begin{aligned} A^3{}_\mu &= \alpha A'^3{}_\mu + \beta A'^0{}_\mu \\ A^0{}_\mu &= \gamma A'^3{}_\mu + \delta A'^0{}_\mu \end{aligned} \tag{28}$$

In order to determine α, β, γ, and δ we impose on (28) the following conditions:

(i) Preservation of the quadratic terms: $(A^0{}_\mu)^2 + (A^3{}_\mu)^2 = (A'^0{}_\mu)^2 + (A'^3{}_\mu)^2$. This condition is imposed as if there were mass terms in the Lagrangian, which we want to preserve.
(ii) Transformation continuously connected to the identity.

This leads to the conditions

$$\alpha\beta + \gamma\delta = 0 \tag{29}$$

$$\alpha^2 + \gamma^2 = 1 \tag{30}$$

$$\beta^2 + \delta^2 = 1 \tag{31}$$

$$\alpha\delta - \gamma\beta = 1 \tag{32}$$

from which result the following cases:

Case (i): $\delta = \alpha$, $\beta = -\gamma \Rightarrow \alpha^2 + \gamma^2 = 1$. For this we choose the parametrization $\alpha = \delta = \cos\theta$, $\quad \beta = -\gamma = \sin\theta$

Case (ii): $\delta = \alpha$, $\beta = \gamma \Rightarrow \alpha^2 = 1$, $\gamma = 0$, $\beta = 0$.

Case (iii): $\delta = -\alpha$, $\beta = -\gamma \Rightarrow \gamma^2 = 1$, $\alpha = 0$, $\delta = 0$.

We write (28) for the three above cases.

Case (i):

$$A^3{}_\mu = \cos\theta\, A'^3{}_\mu + \sin\theta\, A'^0{}_\mu \tag{33}$$
$$A^0{}_\mu = -\sin\theta\, A'^3{}_\mu + \cos\theta\, A'^0{}_\mu$$

Case (ii)

$$A^3{}_\mu = \pm A'^3{}_\mu \tag{34}$$
$$A^0{}_\mu = \pm A'^0{}_\mu$$

Case (iii)

$$A^3{}_\mu = \mp A'^0{}_\mu \tag{35}$$
$$A^0{}_\mu = \pm A'^3{}_\mu$$

Expressions (33) correspond to the usual expressions of mixing gauge fields in Weinberg–Salam theory and we will use them to construct the new algebra.

Considering (19) and (24), we write (25) explicitly

$$A'^1{}_\mu X'_1 + A'^2{}_\mu X'_2 + A'^3{}_\mu X'_3 + A'^0{}_\mu X'_0$$
$$= A^1{}_\mu X_1 + A^2{}_\mu X_2 + A^3{}_\mu X_3 + A^0{}_\mu X_0 \tag{36}$$

Using (26), (27), and (33) and equating the coefficients of each component of $A'^a{}_\mu$, we obtain the new generators

$$X'_1 = \frac{1}{\sqrt{2}}(X_1 + iX_2) \tag{37}$$

$$X'_2 = \frac{1}{\sqrt{2}}(X_1 - iX_2) \tag{38}$$

$$X'_3 = \cos\theta\, X_3 - \sin\theta\, X_0 \tag{39}$$

$$X'_0 = \sin\theta\, X_3 + \cos\theta\, X_0 \tag{40}$$

The new algebra is characterized by the commutation relations of the fields $\{X'_a\}$:

$$\begin{aligned}
[X'_1, X'_2] &= -i(\sin\theta\, X'_0 + \cos\theta\, X'_3)\\
[X'_1, X'_3] &= i\cos\theta\, X'_1\\
[X'_1, X'_0] &= i\sin\theta\, X'_1\\
[X'_2, X'_3] &= -i\cos\theta\, X'_2\\
[X'_2, X'_0] &= -i\sin\theta\, X'_2
\end{aligned} \tag{41}$$

$$[X_0', X_3'] = 0$$

We observe that there is now a mixing between the generators of the two sectors.

Cases (ii) and (iii) above are particular cases of (i). To case (ii) there correspond the angles $\theta = 0$ and $\theta = \pi$, giving rise to the algebra

$$\begin{aligned}
[X_1', X_2'] &= \mp i X_3' \\
[X_1', X_3'] &= \pm i X_1' \\
[X_1', X_0'] &= 0 \\
[X_2', X_3'] &= \mp i X_2' \\
[X_2', X_0'] &= 0 \\
[X_0', X_3'] &= 0
\end{aligned} \tag{42}$$

For case (iii) we have $\theta = \pi/2$ and $\theta = 3\pi/2$ and the algebra is

$$\begin{aligned}
[X_1', X_2'] &= \mp i X_0' \\
[X_1', X_3'] &= 0 \\
[X_1', X_0'] &= \pm i X_1' \\
[X_2', X_3'] &= 0 \\
[X_2', X_0'] &= \mp i X_2' \\
[X_0', X_3'] &= 0
\end{aligned} \tag{43}$$

These three sets of commutation relations satisfy Jacobi identities and therefore constitute Lie algebras. As noticed, the last two algebras above are particular cases of the most general one given by (41), which we call *Glashow algebra* (*GA*).

Since we do not have masses in the theory, all values of angles are admissible and we could be tempted to construct a theory for each one of these algebras. But with mass generation and considering the usual masses relations of the Weinberg–Salam model, in which $\sin\theta$ appears in the denominator, algebras (ii) and (iii) are actually excluded since they would imply infinite masses for the gauge bosons.

The structure constants of the new algebra (41) are

$$f'^{0}{}_{12} = -i \sin\theta$$

$$f'^{3}{}_{12} = -i \cos\theta$$

$$
\begin{aligned}
f'^{1}{}_{13} &= i \cos\theta \\
f'^{1}{}_{10} &= i \sin\theta \\
f'^{2}{}_{23} &= -i \cos\theta \\
f'^{2}{}_{20} &= -i \sin\theta
\end{aligned} \tag{44}
$$

These are quite different from those of the direct product algebra.

The Killing–Cartan bilinear form associated to the group is given by

$$g_{ab} = f^{c}{}_{ad} f^{d}{}_{bc} \tag{45}$$

and its determinant is equal to zero. This characterizes the Glashow algebra as a nonsemisimple one. We have studied other invariant metrics (Aldrovandi *et al.*, 1999), but they all have null determinant.

4. REPRESENTATIONS OF THE GLASHOW ALGEBRA

We now present two matrix representations of the Glashow algebra. The first one to be considered is the adjoint representation which is constructed using the structure constants (44). In that case, the generators are

$$J'_1 = \begin{bmatrix} 0 & 0 & i\cos\theta & i\sin\theta \\ 0 & 0 & 0 & 0 \\ 0 & -i\cos\theta & 0 & 0 \\ 0 & -i\sin\theta & 0 & 0 \end{bmatrix} \tag{46}$$

$$J'_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i\cos\theta & -i\sin\theta \\ i\cos\theta & 0 & 0 & 0 \\ i\sin\theta & 0 & 0 & 0 \end{bmatrix} \tag{47}$$

$$J'_3 = \begin{bmatrix} -i\cos\theta & 0 & 0 & 0 \\ 0 & i\cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

$$J'_0 = \begin{bmatrix} -i\sin\theta & 0 & 0 & 0 \\ 0 & i\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{49}$$

Another representation can be obtained if we consider the adjoint representation of the direct product (15)–(18) and apply to it the transformations (37)–(40):

$$X_1' = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & -1 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{50}$$

$$X_2' = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & -1 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{51}$$

$$X_3' = \begin{bmatrix} 0 & -\cos\theta & 0 & 0 \\ \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sqrt{2}\sin\theta \end{bmatrix} \tag{52}$$

$$X_0' = \begin{bmatrix} 0 & -\sin\theta & 0 & 0 \\ \sin\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{2}\cos\theta \end{bmatrix} \tag{53}$$

These are the transformed matrices of the adjoint representation of the direct product, which is different from the adjoint representation of the transformed algebra. The adjoint representation of the transformed algebra does not coincide with the transformed of the adjoint representation of $SU(2) \otimes U(1)$. This is the representation which will lead to the Lagrangian, as will be seen in Section 5.

There is still another representation for Glashow algebra. It corresponds to the transformed representation of the fundamental representation of the direct product:

$$T_1' = -\frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{54}$$

$$T_2' = -\frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{55}$$

$$T'_3 = \frac{i}{2}\begin{bmatrix} -\cos\theta & 0 & 0 \\ 1 & \cos\theta & 0 \\ 0 & 0 & -\sqrt{2}\sin\theta \end{bmatrix} \tag{56}$$

$$T'_0 = \frac{i}{2}\begin{bmatrix} -\sin\theta & 0 & 0 \\ 1 & \sin\theta & 0 \\ 0 & 0 & \sqrt{2}\cos\theta \end{bmatrix} \tag{57}$$

This is the single 3×3 representation found, and can be considered as the fundamental representation of the Glashow algebra.

Now a natural question arises: are the A'_μ gauge fields for the Glashow algebra, that is, do they belong to the adjoint representation? To answer this question we must determine the behavior of A'_μ under action of the fields X'_a. We use the transformation (20) of the gauge fields A_μ and the expressions (26), (27), and (33) of these fields, as well as (37)–(40), to obtain

$$X'_a\,(A'^b{}_\mu) = f'^b{}_{ca}\,A'^c{}_\mu \tag{58}$$

The structure constants (44) have also been used.

Thus, the fields A'_μ are indeed gauge fields. We may obtain their equations of motion in the usual way for gauge theories, via the duality prescription applied to the Bianchi identity. Or we may consider alternatively the usual Lagrangian of gauge theory to describe their dynamics.

The field strength associated to A'_μ can be determined using the same argument above (25) to write the equality

$$F'^a{}_{\mu\nu}\,X_a = F'^a{}_{\mu\nu}\,X'_a \tag{59}$$

Using the expression of the field strength of A_μ, we have

$$g\{\partial_\mu\,A^a{}_\nu\,X_a - \partial_\nu\,A^a{}_\mu\,X_a + g[A^b{}_\mu\,X_b,\,A^c{}_\nu\,X_c]\} = F'^a{}_{\mu\nu}\,X'_a \tag{60}$$

Now applying (25), we obtain

$$g\{\partial_\mu\,A'^a{}_\nu\,X'_a - \partial_\nu\,A'^a{}_\mu\,X'_a + g[A'^b{}_\mu\,X'_b,\,A'^c{}_\nu\,X'_c]\} = F'^a{}_{\mu\nu}\,X'_a \tag{61}$$

which gives

$$F'^a{}_{\mu\nu} = g[\partial_\mu\,A'^a{}_\nu - \partial_\nu\,A'^a{}_\mu + gf'^a{}_{\mathrm{bc}}\,A'^b{}_\mu\,A'^c{}_\nu] \tag{62}$$

We have proceeded with this punctiliousness because of the above finding, according to which the adjoint of *GA* is not the transformed of the adjoint of $SU(2) \otimes U(1)$.

With the help of (44) these expressions can be explicitly written as

$$F'^1_{\mu\nu} = g[\partial_\mu A'^1_\nu - \partial_\nu A'^1_\mu + ig \cos\theta\, (A'^1_\mu A'^3_\nu - A'^1_\nu A'^3_\mu) + ig \sin\theta (A'^1_\mu A'^0_\nu - A'^1_\nu A'^0_\mu)] \qquad (63)$$

$$F'^2_{\mu\nu} = g[\partial_\mu A'^2_\nu - \partial_\nu A'^2_\mu - ig \cos\theta\, (A'^2_\mu A'^3_\nu - A'^2_\nu A'^3_\mu) - ig \sin\theta (A'^2_\mu A'^0_\nu - A'^2_\nu A'^0_\mu)] \qquad (64)$$

$$F'^3_{\mu\nu} = g[\partial_\mu A'^3_\nu - \partial_\nu A'^3_\mu - ig \cos\theta\, (A'^1_\mu A'^2_\nu - A'^2_\nu A'^1_\mu)] \qquad (65)$$

$$F'^0_{\mu\nu} = g[\partial_\mu A'^0_\nu - \partial_\nu A'^0_\mu - ig \sin\theta\, (A'^1_\mu A'^2_\nu - A'^2_\nu A'^1_\mu)] \qquad (66)$$

Once in possession of the field strength, we can now construct the Lagrangian.

5. LAGRANGIAN

The Lagrangian for a gauge theory is

$$L = \frac{1}{8g^2} \int d^3x\, tr\, (F_{\mu\nu} F^{\mu\nu}) \qquad (67)$$

where in the present case $F_{\mu\nu}$ is the field strength in the original algebra, that is, the direct product algebra. In order to obtain the expression of the Lagrangian in the Glashow case we use (59):

$$L = \frac{1}{8g^2} \int d^3x\, F'^a_{\mu\nu} F'^{b\mu\nu}\, tr\, (X'_a X'_b) \qquad (68)$$

where X'_a are elements of the transformed representation of the adjoint algebra of the direct product (50)–(53). It is important to notice that here one does not consider the adjoint representation of the *Glashow algebra*, but coherently, the other representation, that is, the transformation of the adjoint representation of the direct product. If we choose to calculate the traces in the Lagrangian, the adjoint representation of the Glashow algebra, we obtain a derivative coupling between the fields A_ν and Z_ν, which do not correspond to any vertex of the physical theory.

Thus the role of the second representation is to give the traces for the Lagrangian. The nonnull traces are

$$\begin{aligned} tr(X'_0 X'_0) &= -2 \\ tr(X'_1 X'_2) &= -2 \\ tr(X'_3 X'_3) &= -2 \end{aligned} \qquad (69)$$

and the resulting Lagrangian is

$$L = -\frac{1}{4g^2} \int d^3x \, \{F'^0{}_{\mu\nu} F'^{0\mu\nu} + F'^3{}_{\mu\nu} F'^{3\mu\nu} + 2F'^1{}_{\mu\nu} F'^{2\mu\nu}\} \tag{70}$$

Let us substitute (63)–(66) in the last expression. Making the associations

$$\begin{aligned} A'^1{}_\nu &\to W^-_\nu \\ A'^2{}_\nu &\to W^+_\nu \\ A'^3{}_\nu &\to Z_\nu \\ A'^0{}_\nu &\to A_\nu \end{aligned} \tag{71}$$

we obtain the electroweak theory Lagrangian without mass:

$$\begin{aligned} L = \int d^3x \Big\{ & -\frac{1}{4} \mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{4} W^+_{\mu\nu} W^{-\mu\nu} \\ & + ig \cos\theta[(W^-_\mu W^+_\nu - W^-_\nu W^+_\mu)\, \partial^\mu Z^\nu + (\partial_\mu W^+_\nu - \partial_\nu W^+_\mu) W^{-\nu} Z^\mu \\ & - (\partial_\mu W^-_\nu - \partial_\nu W^-_\mu) W^{+\nu} Z^\mu] \\ & + ig \sin\theta[(W^-_\mu W^+_\nu - W^-_\nu W^+_\mu)\, \partial^\mu A^\nu + (\partial_\mu W^+_\nu - \partial_\nu W^+_\mu) W^{-\nu} A^\mu \\ & - (\partial_\mu W^-_\nu - \partial_\nu W^-_\mu) W^{+\nu} A^\mu] \\ & + g^2 \cos^2\theta[W^+_\mu W^-_\nu Z^\mu Z^\nu - W^+_\mu W^{-\mu} Z_\nu Z^\nu] \\ & + g^2 \sin^2\theta[W^+_\mu W^-_\nu A^\mu A^\nu - W^+_\mu W^{-\mu} A_\nu A^\nu] \\ & + g^2 \sin\theta \cos\theta[W^+_\mu W^-_\nu (Z^\mu A^\nu + A^\mu Z^\nu) - 2W^+_\mu W^{-\mu} A_\nu Z^\nu] \\ & + \frac{1}{2} g^2 W^-_\mu W^+_\nu [W^{-\mu} W^{+\nu} - W^{-\nu} W^{+\mu}] \Big\} \end{aligned} \tag{72}$$

where

$$\mathscr{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{73}$$

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \tag{74}$$

$$W^+_{\mu\nu} = \partial_\mu W^+_\nu - \partial_\nu W^+_\mu \tag{75}$$

$$W^{-\mu\nu} = \partial^\mu W^{-\nu} - \partial^\nu W^{-\mu} \tag{76}$$

6. CONCLUSION

On the bundle of the electroweak theory, we have constructed an algebra *GA* through basis transformations affecting only the fundamental, vertical fields. The mixing angle is incorporated in the *GA* structure constants. *GA* has two representations in terms of 4×4 matrices. One of them is its adjoint representation and the other is the transformed of the adjoint representation of the algebra of $SU(2) \otimes U(1)$. It is an important point that they do not coincide. It is the transformed of the adjoint representation of $SU(2) \otimes U(1)$ which provides the traces leading to the Lagrangian. The result is just the Lagrangian of electroweak interactions without masses.

These massless "physical" fields are the gauge fields written in the new basis. They are indeed gauge potentials for the Glashow algebra, that is, they belong to its adjoint representation.

Every gauge theory has an underlying bundle which fixes its geometrical aspects. The local properties are summed up in the algebra of the vector fields tangent to the bundle. *GA* is that algebra for the Glashow model, a basic step in the construction of the electroweak theory.

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